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EQUILIBRIUM POINTS IN GAMES
WITH VECTOR PAYOFFS

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SUMMARY

The payoff of a game sometimes takes the form of a vector having components that represent amounts of different things, such as ships, men, money, etc., of which the relative values are unknown. The purpose of this paper is to define and characterize the equilibrium-point solutions of games of this kind.

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1. INTRODUCTION

The payoff of a game sometimes most naturally takes the form of a vector having numerical components that represent commodities (such as men, ships, money, etc.) whose relative values cannot be ascertained. The utility spaces of the players can therefore be given only a partial ordering (representable as the intersection of a finite number of total orderings), and the usual notions of solution must be generalized. In this note we define and characterize the noncooperative (equilibrium-point) solutions of such vector games.*

2. DEFINITIONS

Let R^n denote the space of vectors $a = (a^1, \ldots, a^n)$ and define two order relations on R^n:

$$a \;\textcircled{>}\; b \quad \Longleftrightarrow \quad a^k > b^k, \quad \text{all } k,$$

$$a \;\boxed{>}\; b \quad \Longleftrightarrow \quad \begin{cases} a^k \geq b^k, & \text{all } k, \text{ with} \\ a^k > b^k & \text{at least once.} \end{cases}$$

It is an easy matter to verify the following two properties:

$$(1) \qquad a \;\textcircled{>}\; 0 \quad \Longleftrightarrow \quad a\alpha > 0, \quad \text{all } \alpha \;\boxed{>}\; 0,$$

$$(2) \qquad a \;\boxed{>}\; 0 \quad \Longleftrightarrow \quad a\alpha > 0, \quad \text{all } \alpha \;\textcircled{>}\; 0,$$

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*We are indebted to Dr. F. D. Rigby of the Office of Naval Research for several suggestions relating to this problem.

where $a\alpha$ is the inner product $\sum_1^n a^k \alpha^k$ of the vectors a and α.

The game to be considered has a zero-sum payoff matrix $A = (a_{ij})$, each $a_{ij} \in R^n$. It is assumed that the players respond linearly to probability mixtures of vectors, so that the "mixed" outcome consisting of the vectors

$$\{a_{i_\nu j_\nu} \text{ with probability } p_\nu\},\ \nu = 1, \ldots, N,\ \sum_1^N p_\nu = 1,$$

is entirely equivalent to the "pure" outcome $\sum_1^N p_\nu a_{i_\nu j_\nu} \in R^n$. (Compare the "mixture" axioms of [1], [2], or [3].) It is also assumed that the first player wants to increase the components of the vector, and the second player wants to decrease them. Finally, it is assumed that neither player has an a priori opinion concerning the relative importance to himself of the different components. We shall concentrate on the strongest and weakest preference orderings compatible with these hypotheses—namely, $(>)$ and $[>]$ and their inverses—and omit discussion of the numerous intermediate cases.

Since the goals of the players are as directly opposed as possible, considering the incompleteness of their preferences, it is natural to look first at the "noncooperative" types of solution. Writing xAy for $\sum_i \sum_j x_i a_{ij} y_j$, we define the mixed strategy pair (x^*, y^*) to be a <u>strong equilibrium point</u> (SEP) if and only if the vector $v = x^*Ay^*$ is simultaneously <u>maximal</u> in the set

(3) $\quad F = \{xAy^* \mid x \text{ arbitrary}\}$

and minimal in the set

(4) $\quad G = \{x^*Ay \mid y \text{ arbitrary}\}$,

in the sense of the ordering $\boxed{>}$. That is, there is no strategy x for which $xAy^* \boxed{>} x^*Ay^*$ and no strategy y for which $x^*Ay^* \boxed{>} x^*Ay$. Similarly, (x^*, y^*) is a weak equilibrium point (WEP) if v is maximal in (3) and minimal in (4) in the sense of $(>)$. Clearly, every SEP is a WEP.

3. CHARACTERIZATION OF EQUILIBRIUM POINTS

We wish to find a characterization of the SEP and WEP just defined, in terms of ordinary equilibrium-point theory [5]. First let us introduce a set of nonnegative "weighting factors" $\alpha = (\alpha^1, \ldots, \alpha^n)$, and consider the resulting numerical game $\alpha A = (\alpha a_{ij})$. If x^*, y^* are optimal mixed strategies in this game, and if $v = x^*Ay^*$, then αv is its value, and we have

(5) $\quad \alpha(xAy^* - v) \leq 0, \qquad$ all x,

(6) $\quad \alpha(v - x^*Ay) \leq 0, \qquad$ all y.

Now, if $\alpha \boxed{>} 0$, then $xAy^* - v$ is never $(>)$ 0, by (1) or (2); likewise $v - x^*Ay \not{(>)}$ 0. Thus, (x^*, y^*) is a WEP. In similar fashion, if $\alpha (>) 0$, then (x^*, y^*) is a SEP.

This shows that equilibria of both types always exist, and provides a simple way of finding some of them—namely, by solving certain zero-sum numerical games of the form αA. However, to find them all we must assign different weighting factors, $\alpha = (\alpha^1, \ldots, \alpha^n)$, $\beta = (\beta^1, \ldots, \beta^n)$, to the two

players' payoffs. This leads to a non-zero-sum numerical game, described by the pair of matrices $[\alpha A, -\beta A]$. This game, of course, has equilibrium points of the ordinary kind—namely, pairs (x^*, y^*) such that

(7) $\qquad x^*(\alpha A)y^* \geq x(\alpha A)y^*, \qquad$ all x,

(8) $\qquad x^*(\beta A)y^* \leq x^*(\beta A)y, \qquad$ all y.

From these, expressions similar to (5) and (6) can be derived, and we see as before that (x^*, y^*) is a WEP of the vector game if $\alpha \boxed{>} 0$, $\beta \boxed{>} 0$, and is a SEP if $\alpha \textcircled{>} 0$, $\beta \textcircled{>} 0$.

It remains to show that all WEP and SEP of the vector game A can be obtained from the games $[\alpha A, -\beta A]$ by the above method. (Our original approach is, of course, included, via the special case $\alpha = \beta$.) Let (x^*, y^*) be a WEP of A, let $v = x^*Ay^*$, and let F and G be the convex sets in R^n defined by (3) and (4). Let $\mathcal{E}(\textcircled{>}, F)$ be the extension of F obtained by including all vectors b such that $a \textcircled{>} b$ for some $a \in F$. The extended set is still convex, and has the same maximal vectors (in the sense of $\textcircled{>}$) as F. Hence, v is maximal in $\mathcal{E}(\textcircled{>}, F)$, and lies in its boundary. Consider any hyperplane that supports $\mathcal{E}(\textcircled{>}, F)$ at v; if α is its (outward-pointing) normal, then we have

(9) $\qquad \alpha(v - a) \geq 0, \qquad$ all $a \in \mathcal{E}(\textcircled{>}, F)$.

In particular, we have $\alpha(v - a) > 0$ whenever $v \circledgt a$, since the set of vectors so dominated by v is open and contained in $\mathcal{E}(\circledgt, F)$. This implies that $\alpha \boxed{>} 0$, by (2). A similar construction determines a $\beta \boxed{>} 0$ such that

$$(10) \qquad \beta(v - b) \leq 0, \qquad \text{all } b \in \mathcal{E}(\circledlt, G).$$

But F is contained in $\mathcal{E}(\circledgt, F)$; hence (9) with (3) reduces to (7). Similarly (10) and (4) give (8). This shows that our WEP (x^*, y^*) is an equilibrium point of the numerical general-sum game $[\alpha A, -\beta A]$, with $\alpha \boxed{>} 0$, $\beta \boxed{>} 0$.

The corresponding proof for SEP is more complicated. Let (x^*, y^*) be a SEP of A and construct the extended set $\mathcal{E}(\boxed{>}, F)$. Let C^r be the r-dimensional set obtained by intersecting $\mathcal{E}(\boxed{>}, F)$ with <u>all</u> supporting hyperplanes at v. Since $\mathcal{E}(\boxed{>}, F)$ is closed and polyhedral,* in addition to being convex, it follows that C^r is the closed boundary face of lowest dimension containing v, and that v is in the (relative) interior of C^r. We can find a supporting hyperplane whose intersection with $\mathcal{E}(\boxed{>}, F)$ is precisely C^r (any interior member of the set of all supporting hyperplanes at v will do). Then, if α is its normal we have

$$\alpha(v - a) \geq 0, \qquad \text{all } a \in \mathcal{E}(\boxed{>}, F),$$

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*It is essential here that $\mathcal{E}(\boxed{>}, F)$ be polyhedral (see Chapter 1 of [4], especially pp. 5-8). In fact, a supporting hyperplane with strictly positive normal does not exist everywhere, if (say) F is a sphere.

with equality only if $a \in C^r$.

We claim that $\alpha(v - a) > 0$ whenever $v \boxed{>} a$. If not, there would be a vector $a \in C^r$ such that $v \boxed{>} a$. But, since v is in the relative interior of C^r, there would also be a vector b (of the form $b = (1 + \epsilon)v - \epsilon a$, $\epsilon > 0$) in C^r such that $b \boxed{>} v$. This contradicts the maximality of v. Thus, we conclude by (1) that $\alpha > 0$. We can now proceed as in the previous proof. We sum up:

<u>Theorem.</u> <u>The WEP of the zero-sum vector game A are precisely the equilibrium points of the general-sum, numerical games $[\alpha A, -\beta A]$ with $\alpha \boxed{>} 0$, $\beta \boxed{>} 0$. The SEP of A are precisely the equilibrium points of the games $[\alpha A, -\beta A]$ with</u> $\alpha > 0$, $\beta > 0$.

4. EXTENSIONS TO INFINITE AND NON-ZERO-SUM GAMES

In vector games with infinitely many strategies the characterization of WEP is exactly analogous; but the characterization of SEP is more complex, since the sets F, G are not necessarily polyhedral (see the footnote).

A similar result holds for non-zero-sum vector games. In this case it is not even necessary that the payoffs of the two players be vectors in the same space. Thus, the WEP of $[A, B]$, where $a_{ij} \in R^m$, $b_{ij} \in R^n$, are precisely the equilibrium points of $[\alpha A, \beta B]$, where $\alpha \in R^m$, $\beta \in R^n$, and $\alpha \boxed{>} 0$, $\beta \boxed{>} 0$, etc. The arguments in favor of the noncooperative type of solution, however, are less compelling for these games than before, in the zero-sum case.

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